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THE MATHEMATICAL THEORY OF THE TOP

BY A. G. GREENHILL

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17. The dynamical constants, G , G' , E , being three in number, there is a triply infinite (∞^3) assemblage of states of motion of the top, so that the selection of an illustrative numerical example is at first sight an embarrassing one in its variety.

But the student requires some well selected numerical examples of an actual case of motion, to fix his ideas and to refer to for elucidation, and this can be obtained by choosing for the parameter v in (13) §9 a simple aliquot part of a period, so that f is a rational fraction; in this way we can utilize Abel's theory, and replace the theta functions in the general integral $I(v)$ of (1) and (2) §10 by algebraical functions.

Beginning with the simplest case, take

$$(1) \quad GE - G' = 0, \quad Q = 0;$$

then v is a half-period, either ω_1 or ω_2 but never ω_3 ; $f = 0$ or 1 ;

$E = z_1$ or z_2 , never z_3 for real motion, because $E - z$ must be positive;

$$(2) \quad x_1 = 0 \quad \text{or} \quad x_2 = 0, \quad \rho_1 = 0 \quad \text{or} \quad \rho_2 = 0,$$

$$(3) \quad \rho = \rho_3 \operatorname{dn}(K - mt), \quad \text{or} \quad \rho_3 \operatorname{cn}(K - mt),$$

and replacing the k of § 7 by the letter c , to avoid confusion with the modulus of the elliptic functions,

$$(4) \quad \omega = \frac{\delta}{c} nt = \frac{L}{M} nt = pt,$$

suppose, giving the curve of H ; and in the associated motion of the top

$$(5) \quad \chi = pt - \psi,$$

$$(6) \quad \begin{aligned} \frac{1}{2} M^2 \sin \theta e^{\chi i} &= L \sqrt{(x - x_1) + i \sqrt{(x_3 - x)(x - x_2)}}, \\ \text{or} \quad &L \sqrt{(x - x_2) + i \sqrt{(x_3 - x)(x - x_1)}}; \end{aligned}$$

$$(7) \quad \sin \theta \sin(pt - \psi) = \sqrt{(z_2 - z)(z - z_3)} \quad \text{or} \quad \sqrt{(z_1 - z)(z - z_3)},$$

$$(8) \quad \sin \theta \cos(pt - \psi) = \frac{L}{M} \sqrt{2(z_1 - z)}, \quad \text{or} \quad \frac{L}{M} \sqrt{2(z_2 - z)},$$

$$(9) \quad \frac{p}{n} = \frac{L}{M} = \sqrt{\frac{z_2 + z_3}{2}}, \quad \text{or} \quad \sqrt{\frac{z_1 + z_3}{2}},$$

$$(10) \quad \frac{(\tau')}{(\tau)} = z_1 = \frac{1 + z_2 z_3}{z_2 + z_3}, \quad \text{or} \quad \frac{(\tau')}{(\tau)} = z_2 = \frac{1 + z_1 z_3}{z_1 + z_3}.$$

18. Next take the bisection of a half-period where $f = \frac{1}{2}$:

$$(1) \quad r = \omega_1 + \frac{1}{2}\omega_3, \quad \text{or} \quad H' + \frac{1}{2}K'i,$$

and work with

$$(2) \quad I = \frac{1}{2} \int_x^k \frac{(1-k)(k+x^2)}{k-x^2} \frac{dx}{\sqrt{X}},$$

$$(3) \quad X = (1-x^2)(k^2-x^2), \quad P = \frac{1}{2}(1-k), \quad Q = k(1-k),$$

$$(4) \quad pt - \omega = I = \cos^{-1} \sqrt{\frac{(1-x)(k+x)}{2(k-x^2)}} = \sin^{-1} \sqrt{\frac{(1+x)(k-x)}{2(k-x^2)}},$$

$$(5) \quad nt = \int_x^k \frac{M dx}{\sqrt{X}}, \quad \frac{p}{n} = \frac{L - \frac{1}{2}(1-k)}{M},$$

$$(6) \quad x = k \operatorname{sn}(H' - mt), \quad n = Mn;$$

and the curve of II is given by

$$(7) \quad \frac{\rho}{c} e^{(pt-\omega)i} = \frac{\sqrt{\frac{1}{2}(1-x)(k+x)} + i\sqrt{\frac{1}{2}(1+x)(k-x)}}{M},$$

$$(8) \quad \frac{\rho^2}{c^2} = \frac{k-x^2}{M^2},$$

$$(9) \quad e^{(pt-\omega)i} = \frac{\sqrt{\frac{1}{2}(1-x)(k+x)} + i\sqrt{\frac{1}{2}(1+x)(k-x)}}{\sqrt{(k-x^2)}},$$

$$(10) \quad \begin{aligned} \sin 2(pt - \omega) &= \frac{\operatorname{cn}(H' - mt) \operatorname{dn}(H' - mt)}{1 - k \operatorname{sn}^2(H' - mt)} \\ &= \frac{(1+k) \operatorname{sn} mt}{1 + k \operatorname{sn}^2 mt} \end{aligned}$$

$$= \operatorname{sn}[(1+k)mt, \gamma], \quad \gamma = \frac{2\sqrt{k}}{1+k},$$

by Landen's Second Transformation:

$$(11) \quad pt - \omega = \frac{1}{2} \operatorname{am}[(1+k)nt, \gamma].$$

In the associated motion of the top, derived in the preceding manner,

$$(12) \quad \frac{1}{2} M^2 \sin \theta e^{(pt - \psi)i} = (L - 1 + k - x) \sqrt{\frac{1}{2}(1 - x)(k + x)} \\ + i(L - 1 + k + x) \sqrt{\frac{1}{2}(1 + x)(k - x)},$$

$$(13) \quad M^2 \cos \theta = L^2 - 1 + k - k^2 + 2x^2,$$

$$(14) \quad (\frac{1}{2} M^2 \sin \theta)^2 = k(L - 1 + k)^2 - (L^2 - 1 + k - k^2)x^2 - x^4,$$

$$(15) \quad M^2 z_1 = L^2 + 1 + k - k^2, \quad \frac{1}{2} M^2 \operatorname{sh} \theta_1 = (L + k) \sqrt{(1 - k)},$$

$$(16) \quad M^2 z_2 = L^2 - 1 + k + k^2, \quad \frac{1}{2} M^2 \sin \theta_2 = (L - 1) \sqrt{(k - k^2)},$$

$$(17) \quad M^2 z_3 = L^2 - 1 + k - k^2, \quad \frac{1}{2} M^2 \sin \theta_3 = (L - 1 + k) \sqrt{k};$$

$$(18) \quad M^2 E = L^2 - 1 + 3k - k^2,$$

$$(19) \quad M^4 = (L^2 - 1 + 3k - k^2)^2 - 8k(1 - k)L + 8k(1 - k)^2.$$

$$(20) \quad L' = \frac{L^3 - (1 - 3k + k^2)L - 2k(1 - k)}{M^2}.$$

Thus $L' = L = -k$ gives an intermediate path, and we find

$$(21) \quad pt - \psi = \frac{3}{4} \pi + \frac{1}{2} \operatorname{am} [\Gamma - (1 + k) mt, \gamma];$$

$L' = L = 1$ gives an upper rosette, with

$$(22) \quad pt - \psi = -\frac{1}{4} \pi + \frac{1}{2} \operatorname{am} [\Gamma - (1 + k) mt, \gamma],$$

and $L' = -L = -1 + k$ a lower rosette, with

$$(23) \quad pt - \psi = \pi - \frac{1}{2} \operatorname{am} [(1 + k) mt, \gamma];$$

these rosette curves are shown in a stereoscopic diagram in *Engineering*, 10 Oct., 1897, for $k = 0.77384$, in the upper rosette, and $k = 0.42893$ in the lower, thus making the apsidal angle $\frac{4}{5} \pi$ and $\frac{2}{5} \pi$.

Cusps occur on the circle z_2 for

$$(24) \quad L = 1 \pm \sqrt{(k + k^2)}.$$

The axis moves like a spherical pendulum when $L' = 0$; and putting

$$(25) \quad L = c(1 - k),$$

$$(26) \quad \frac{1}{k} + k = \frac{2c^3 - 3c + 2}{c^3 - c}, \quad \frac{2\sqrt{k}}{1 + k} = 2 \sqrt{\frac{c(c^2 - 1)}{(2c - 1)(2c^2 + c - 2)}}.$$

in which c may be taken as the arbitrary constant for the construction of numerical cases. (*Proc. Lond. Math. Soc.*,* vol. 27, p. 610.)

The motion is purely algebraical when

$$(27) \quad p = 0, \quad L = \frac{1}{2} (1 - k),$$

$$(28) \quad (2M)^4 = (1 + k)^2 (9 - 14k + 9k^2),$$

$$(29) \quad (2M)^2 z_1 = (1 + k) (5 - 3k),$$

$$(30) \quad (2M)^2 z_2 = (1 + k) (-3 + 5k),$$

$$(31) \quad (2M)^2 z_3 = -3 + 2k - 3k^2,$$

$$(32) \quad (2M)^2 E = -3 + 10k - 3k^2,$$

$$(33) \quad L' = -\frac{3(1 + k)^2(1 - k)}{8M^2}.$$

Stereoscopic figures of this case, drawn by Mr. T. I. Dewar, are given in L. M. S., vol. 27, p. 587, drawn for

$$(34) \quad k = \frac{15}{17}, \quad \frac{3}{5}, \quad \text{and} \quad \frac{1}{3} \text{ (cusps)}$$

We take $k = \frac{3}{5}$ as affording a good illustrative diagram of the general case; then

$$(35) \quad L = \frac{1}{5}, \quad L' = -\frac{\sqrt{6}}{5}, \quad M = \frac{2\sqrt[4]{24}}{5}, \quad D = \frac{\sqrt{6}}{24}, \quad E = \frac{\sqrt{6}}{4}, \quad F = \frac{7\sqrt{6}}{24},$$

$$(36) \quad z_1 = \frac{2\sqrt{6}}{3}, \quad z_2 = 0, \quad \theta_2 = 90^\circ, \quad z_3 = -\frac{3}{8}\sqrt{6} = \cos 156^\circ 44'; \quad \sin \theta_3 = \frac{\sqrt{10}}{8},$$

and the curve of H and projection P of C' are shown in figure 5 (at the end of the article), and can be realized experimentally with the bicycle wheel, suspended from a bracket, as shown in figures 3 and 4 (page 19).

In this case the points of half time and of inflexion coincide at h , but this is accidental; the representative points H, K, P are taken for

$$x = \frac{1}{10}; \quad \text{and then } mt = \frac{9}{10}K \text{ approximately.}$$

19. For the trisection of a period, where

$$(1) \quad v = \omega_1 + f\omega_3, \quad f = \frac{1}{3} \text{ or } \frac{2}{3},$$

* Abbreviated to L. M. S.

we can utilize the forms given in L. M. S., vol. 25, p. 218, for $\mu = 6$; and changing c there into $\frac{b-1}{b+1}$, put

$$(2) \quad P = \frac{1}{6}(9-b^2), \quad Q = \frac{1}{2}(9-b^2)(b^2-1),$$

$$(3) \quad x_2 = \frac{1}{4}(9-b^2)(b^2-1), \quad x_3 = (3+b)(b-1), \quad x_1 = -(3-b)(b+1),$$

$$(4) \quad \begin{aligned} I &= \int_{x_2}^x \frac{-\frac{1}{6}(9-b^2)x + \frac{1}{2}(9-b^2)(b^2-1)}{x} \frac{dx}{\sqrt{X}} \\ &= \frac{1}{3} \cos^{-1} \frac{\frac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)}}{x^{\frac{3}{2}}} \\ &= \frac{1}{3} \sin^{-1} \frac{(x+9-b^2)\sqrt{(x-x_2)}}{x^{\frac{3}{2}}}; \end{aligned}$$

so that the curve of II is given by

$$(5) \quad M \frac{p}{c} e^{(pt-\omega)i} = \left\{ \frac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)} + i(x+9-b^2)\sqrt{(x-x_2)} \right\}^{\frac{1}{2}},$$

$$(6) \quad \frac{p}{n} = \frac{L - \frac{1}{6}(9-b^2)}{M}.$$

The associated motion of the top is given by equation (3) §13,

$$(7) \quad \frac{1}{2}M^2 \sin \theta e^{(pt-\psi)i} = \frac{Lx - \frac{1}{2}(9-b^2)(b^2-1) + \frac{1}{2}i\sqrt{X}}{\sqrt{x}} \\ \times \left\{ \frac{\frac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)} + i(x+9-b^2)\sqrt{(x-x_2)}}{x^{\frac{3}{2}}} \right\}^{\frac{1}{2}},$$

$$(8) \quad M^2 \cos \theta = L^2 - \frac{1}{4}(9-b^2)^2 + 12 - 2x,$$

$$(9) \quad M^4 = \frac{1}{4}L^2 - \frac{1}{4}(9-b^2)^2 + 12x^2 - 4(9-b^2)(b^2-1)L \\ - 2(9-b^2)(b^2-1)(b^2-5),$$

and working this out we find

$$(10) \quad \left(\frac{1}{2}M^2 \sin \theta\right)^3 e^{3ipt-\psi i} = R\sqrt{(x_3-x)(x-x_1)} + iS\sqrt{(x-x_2)},$$

$$(11) \quad \left(\frac{1}{2}M^2 \sin \theta\right)^2 = -x^2 + \frac{1}{4}L^2 - \frac{1}{4}(9-b^2)^2 + 12x - (9-b^2)(b^2-1)L \\ - \frac{1}{2}(9-b^2)(b^2-1)(b^2-5),$$

$$(12) \quad R = -x^2 - \frac{1}{4}3L^2 - \frac{3}{2}(9-b^2)L + \frac{1}{2}(b^2-3)(b^2-13)x \\ + \frac{1}{2}(9-b^2)\left[L + \frac{1}{2}(b^2-5)\right]^3 - (b^2-1)^2,$$

$$(13) \quad S = [3L - \frac{1}{2}(9 - b^2)]x^2 + \{L^3 - \frac{3}{2}(9 - b^2)L^2 + \frac{3}{4}(b^4 - 22b^2 + 69)L \\ + \frac{1}{8}(9 - b^2)^2(b^2 - 5)\}x + (9 - b^2)\{[L + \frac{1}{2}(b^2 - 5)]^3 + (b^2 - 1)^2\},$$

a factor x^3 cancelling.

Arranged in powers of L ,

$$(14) \quad R = \frac{1}{2}(9 - b^2)L^3 - 3\{x - \frac{1}{4}(9 - b^2)(b^2 - 5)\}L^2 \\ + \frac{3}{2}(9 - b^2)\{x + \frac{1}{4}(b^2 - 5)^2\}L \\ - x^2 - \frac{1}{2}(b^2 - 3)(b^2 - 13)x + \frac{1}{8}(9 - b^2)\{\frac{1}{8}(b^2 - 5)^3 - (b^2 - 1)^2\},$$

$$(15) \quad S = (x + 9 - b^2)L^3 - \frac{3}{2}(9 - b^2)(x - b^2 + 5)L^2 \\ + 3\{x^2 + \frac{1}{4}(b^4 - 22b^2 + 69)x + \frac{1}{4}(9 - b^2)(b^2 - 5)^2\}L \\ + (9 - b^2)\{-\frac{1}{2}x^2 + \frac{1}{8}(9 - b^2)(b^2 - 5)x + \frac{1}{8}(b^2 - 5)^3 + (b^2 - 1)^2\}.$$

In the upper rosette

$$(16) \quad L = \frac{Q}{x_2} = 2, \quad M = \frac{1}{2}(b^2 - 1),$$

$$(17) \quad (M \cos \frac{1}{2}\theta)^3 e^{3(pt - \psi)i} = \frac{1}{2}\sqrt{X} + i\{\frac{1}{2}(b^2 + 3)x - \frac{1}{2}(b^2 - 1)(b^2 + 7)\},$$

$$(18) \quad \frac{p}{n} = \frac{b^2 + 3}{6M} = \frac{b^2 + 3}{3(b^2 - 1)}.$$

In the lower rosette

$$(19) \quad L = \frac{Q}{x_3} = \frac{1}{2}(3 - b)(b + 1), \quad M^2 = b(b - 1)^2,$$

$$(20) \quad (M \sin \frac{1}{2}\theta)^3 e^{3(pt - \psi)i} = [x - \frac{1}{2}(3 - b)(b - 1)(b^2 + b + 2)]\sqrt{(x - x_1)} \\ + i(3b - b^2)\sqrt{(x_3 - x)(x - x_2)},$$

$$(21) \quad \frac{p}{n} = \frac{3b - b^2}{3M}.$$

In the intermediate orbit

$$(22) \quad L = \frac{Q}{x_1} = -\frac{1}{3}(3 + b)(b - 1), \quad M^2 = b(b + 1)^2,$$

$$(23) \quad (M \cos \frac{1}{2}\theta)^3 e^{3(pt - \psi)i} = [-x - \frac{1}{2}(3 + b)(b + 1)(b^2 - b + 2)]\sqrt{(x_3 - x)} \\ - i(3b + b^2)\sqrt{(x - x_2)(x - x_1)},$$

$$(24) \quad \frac{p}{n} = \frac{-3b - b^2}{3M}.$$

The axis moves like a spherical pendulum when

$$(25) \quad L' = 0, \quad L^3 - \frac{1}{4}(9 - b^2)^2 L + 12L - (9 - b^2)(b^2 - 1) = 0,$$

a quadratic for b^2 in terms of L , so that L may be taken as the parameter.

The conditions (3) and (8) §16 for cusps on $z = z_2$ and $z = z_1$ become

$$(26) \quad (L - 2)^2 = \frac{1}{4}(b^2 - 1)^2, \quad L = \frac{5 - b^2}{2}, \quad \frac{3 + b^2}{2};$$

$$(27) \quad [L + \frac{1}{2}(3 + b)(b - 1)]^2 = b(b + 1)^2.$$

The secular term pt is cancelled by putting

$$(28) \quad L = \frac{1}{6}(9 - b^2);$$

specimen cases of this nature in stereoscopic representation are given in L. M. S., vol. 27, pp. 592, 593, having six branches for the region $3 > b > 1$, and three branches for the region $\infty > b > 3$.

In the algebraical case of six cusps,

$$(29) \quad b^2 = 3, \quad L = 1, \quad M^4 = 16, \quad L' = -2,$$

$$(30) \quad x_3 = 2\sqrt{3}, \quad x_2 = 3, \quad x_1 = -2\sqrt{3}, \quad A = 3, \quad B = -12,$$

$$(31) \quad z_1 = 1 + \sqrt{3}, \quad z_2 = -\frac{1}{2} = \cos 120^\circ, \quad z_3 = 1 - \sqrt{3} = \cos 137^\circ,$$

$$(32) \quad \cos \theta = 1 - \frac{1}{2}x, \quad \sin^2 \theta = x - \frac{1}{4}x^2,$$

$$(33) \quad \sin^3 \theta e^{3\psi i} = \frac{1}{8}(-x^2 + 6x - 12)\sqrt{(-x^2 + 12)} + i(x - 3)^{\frac{3}{2}}.$$

With the other value for L in (26) for cusps

$$(34) \quad b^2 = 3, \quad L = 3, \quad M^4 = 48, \quad L' = 2\sqrt{3},$$

$$(35) \quad z_1 = \sqrt{3} + 1, \quad z_2 = \frac{1}{2}\sqrt{3} = \cos 30^\circ, \quad z_3 = \sqrt{3} - 1 = \cos 43^\circ,$$

$$(36) \quad \cos \theta = \sqrt{3} - \frac{x}{2\sqrt{3}}, \quad 12 \sin^2 \theta = -x^2 + 12x - 24,$$

$$(37) \quad (2\sqrt{3} \sin \theta)^3 e^{\frac{3}{2}(\frac{mt}{\sqrt{3}} - \psi)} = (-x^2 + 12)^{\frac{3}{2}} + 6i(x^2 - 6x + 12)\sqrt{(x - 3)},$$

an illustrative numerical case of quasi-algebraical motion, with cusps at angular interval of about 79° .

For the algebraical case of three cusps we obtain from (27)

$$(38) \quad L + \frac{1}{2}(3 + b)(b - 1) = \frac{1}{3}(b^2 + 3b) = (b + 1)b^{\frac{1}{2}},$$

$$(39) \quad (b^{\frac{1}{2}} - 1)^3 - 2 = 0, \quad b = (\sqrt[3]{2} + 1)^2, \quad L = -\sqrt[3]{4} - \sqrt[3]{2}.$$

In the region $3 > b > 1$,

$$(40) \quad x_3 > x > x_2 > 0 > x_1,$$

$$(41) \quad k^2 = \frac{x_3 - x_2}{x_3 - x_1} = \frac{(3+b)(b-1)^3}{16b}, \quad k'^2 = \frac{(3-b)(b+1)^3}{16b},$$

$$(42) \quad \operatorname{sn} \frac{1}{3} K' = \frac{2}{b+1}, \quad \operatorname{cn} \frac{2}{3} K' = \frac{b-1}{b+1},$$

$$(43) \quad \operatorname{sn} \frac{1}{3} K' + \operatorname{cn} \frac{2}{3} K' = 1.$$

20. Generally for a parameter

$$(1) \quad v = \omega_1 + f\omega_3, \quad \text{or} \quad K + fK', \quad f = \frac{v}{2n+1},$$

obtained by the division of a half-period by an odd number $2n+1$, we must proceed to the problem of Division and Transformation of order $\mu = 4n+2$, as explained in L. M. S. vols. 25 and 27, to discover the relations required for

$$\begin{aligned} (2) \quad I(v) &= \int_{x_2}^x \frac{Px + Q}{x} \frac{dx}{\sqrt{X}} \\ &= \frac{1}{2n+1} \cos^{-1} \frac{P_1 x^{n+1} + P_2 x^{n+2} + \dots + P_n}{x^{n+1}} \sqrt{(x_3 - x)(x - x_1)} \\ &= \frac{1}{2n+1} \sin^{-1} \frac{x^n + Q_1 x^{n-1} + Q_2 x^{n-2} + \dots + Q_n}{x^{n+1}} \sqrt{(x - x_2)}. \end{aligned}$$

A differentiation shows that

$$(3) \quad P_1 = (2n+1)P, \quad \text{and} \quad \frac{P}{\sqrt{(x_3 - x_1)}} = \operatorname{zn} f K',$$

from (4) §10; and for the associated elliptic functions,

$$(4) \quad k^2 = \frac{x_3 - x_2}{x_3 - x_1}, \quad k'^2 = \frac{x_2 - x_1}{x_3 - x_1};$$

while for the Division-Values to the modulus k' ,

$$(5) \quad \operatorname{sn}^2 f K' = \frac{x_1}{x_2 - x_1}, \quad \operatorname{cn}^2 f K' = \frac{x_2}{x_2 - x_1}, \quad \operatorname{dn}^2 f K' = \frac{x_3}{x_3 - x_1};$$

$$(6) \quad \operatorname{sn}^2(1-f) K' = \frac{x_2}{x_3} \cdot \frac{x_3 - x_1}{x_2 - x_1}, \quad \operatorname{cn}^2(1-f) K' = \frac{x_1}{x_3} \cdot \frac{x_3 - x_2}{x_2 - x_1},$$

$$\operatorname{dn}^2(1-f) K' = 1 - \frac{x_2}{x_3}.$$

The curve of H is now given by

$$(7) \quad M \frac{\rho}{c} e^{(\rho t - \omega)t} = \frac{\theta(0) \theta(u - v)}{\lambda \theta u \theta v} \\ = \left\{ E \sqrt{(x_3 - x)(x - x_1)} + i F \sqrt{(x - x_2)} \right\}^{\frac{1}{2n+1}}$$

$$(8) \quad E = P_1 x^{n-1} + P_2 x^{n-2} + \dots + P_n,$$

$$(9) \quad F = x^n + Q_1 x^{n-1} + Q_2 x^{n-2} + \dots + Q_n;$$

and then for the associated motion of the axis of the top

$$(10) \quad \frac{1}{2} M^2 \sin \theta e^{(\rho t - \psi)t} = \frac{Lx - Q + \frac{1}{2} i \sqrt{X}}{\sqrt{x}} \\ \times \left\{ \frac{E \sqrt{(x_3 - x)(x - x_1)} + i F(x - x_2)}{x^{n+\frac{1}{2}}} \right\}^{\frac{1}{2n+1}}$$

21. Rationalizing by raising to the power $2n + 1$, and putting

$$(1) \quad (Lx - Q + \frac{1}{2} i \sqrt{X})^{2n+1} = H + K \sqrt{T},$$

where

$$(2) \quad T = -\frac{1}{4} X = x^3 - Ax^2 + Bx + Q^2,$$

$$(3) \quad H = (2n + 1)(Lx - Q) T^n \\ + \frac{(2n + 1)2n(2n - 1)}{3!} (Lx - Q)^3 T^{n-1} \\ + \frac{(2n + 1)2n(2n - 1)(2n - 2)(2n - 3)}{5!} (Lx - Q)^5 T^{n-2} + \dots \\ = (2n + 1) Lx^{3n+1} + (2n + 1) \left(\frac{2n(2n - 1)}{3!} L^3 - nAL - Q \right) x^{3n} \\ + H_1 x^{3n-1} + \dots + H_{3n};$$

$$(4) \quad K = T^n + \frac{(2n + 1)2n}{2!} (Lx - Q)^2 T^{n-1} + \dots$$

$$= x^{3n} + K_1 x^{3n-1} + \dots + K_{3n}.$$

Then

$$(5) \quad \left(\frac{1}{2} M^2 \sin \theta \right)^{2n+1} e^{(2n+1)(\rho t - \psi)t} = R \sqrt{(x_3 - x)(x - x_1)} + iS \sqrt{(x - x_2)},$$

where

$$(6) \quad \begin{aligned} Rx^{2n+1} &= EH - F(x - x_2)K \\ &= -x^{4n+1} + R_1x^{4n} + \dots + R_{2n}x^{2n+1} + 0; \end{aligned}$$

$$(7) \quad \begin{aligned} Sx^{2n+1} &= FH - E(x - x_3)(x - x_1)K \\ &= S_0x^{4n+1} + S_1x^{4n} + \dots + S_{2n}x^{2n+1} + 0; \end{aligned}$$

the coefficient of powers of x below the $(2n + 1)$ th cancelling.

The calculation of $R_1, R_2, \dots, R_{2n}, S_0, S_1, S_2, \dots, S_{2n}$, is thus a piece of heavy algebraical work, when once E and F have been determined.

A few of the leading coefficients in the general case will serve for verification with the simplest values of the divisor μ ; thus

$$(8) \quad S_0 = (2n + 1)L - P_1 = (2n + 1)(L - P),$$

and

$$(9) \quad pt = (L - P) \int_{x_2}^x \frac{dx}{\sqrt{X}},$$

so that S_0 vanishes in the purely algebraical case where the secular term pt is cancelled by taking

$$(10) \quad L = P.$$

Proceeding with the calculations,

$$(11) \quad R_1 = -(2n + 1)nL^2 + (2n + 1)P_1L - Q_1 + nA + x_2,$$

$$(12) \quad \begin{aligned} S_1 &= \frac{(2n + 1)2n(2n - 1)}{3!}L^3 - (2n + 1)nP_1L^2 \\ &\quad + (2n + 1)(Q_1 - nA)L - (2n + 1)Q + (nA + x_1 + x_3)P_1 - P_2. \end{aligned}$$

$$(13) \quad R_2 = \dots, \quad S_2 = \dots$$

22. Otherwise, arranging H and K in ascending powers of T and descending powers of $Lx - Q$, and writing

$$(1) \quad E(x - x_3)(x - x_1) = E', \quad F(x - x_2) = F'$$

$$(2) \quad Rx^{2n+1} = EH - F'K, \quad Sx^{2n+1} = FH - E'K,$$

then in descending powers of Lx ,

$$(3) \quad Rx^{2n+1} = E(Lx)^{2n+1} - (2n+1) r_1 (Lx)^{2n} \\ + \frac{(2n+1)2n}{2!} r_2 (Lx)^{2n-1} - \dots,$$

$$(4) \quad Sx_{2n}^{2n+1} = F(Lx)^{2n+1} - (2n+1) s_1 (Lx)^{2n} \\ + \frac{(2n+1)2n}{2!} s_2 (Lx)^{2n-1} - \dots,$$

$$(5) \quad r_1 = QE + F', \quad s_1 = QF + E',$$

$$(6) \quad r_2 = (T + Q^2)E + 2QF', \quad s_2 = (T + Q^2)F + 2QE',$$

$$(7) \quad r_3 = (3QT + Q^3)E + (T + 3Q^2)F', \\ s_3 = (3QT + Q^3)F + (T + 3Q^2)E',$$

$$(8) \quad r_4 = (T^2 + 6Q^2T + Q^4)E + (4QT + 4Q^3)F', \\ s_4 = (T^2 + 6Q^2T + Q^4)F + (4QT + 4Q^3)E',$$

$$(9) \quad r_5 = (5QT^2 + 10Q^3T + Q^5)E + (T^2 + 10Q^2T + 5Q^4)F', \\ s_5 = (5QT^2 + 10Q^3T + Q^5)F + (T^2 + 10Q^2T + 5Q^4)E',$$

and so on; and r_1, s_1 have a factor x ; r_n, s_n have a factor x^n .

23. Another way of determining the coefficients in R and S is to take

$$(1) \quad pt - \psi = \frac{1}{2n+1} \cos^{-1} \frac{R\sqrt{(x_3-x)(x-x_1)}}{\{-x^2 + (L^2 + A)x - 2LQ - B\}^{n+\frac{1}{2}}} \\ = \frac{1}{2n+1} \sin^{-1} \frac{S\sqrt{(x-x_2)}}{\{-x^2 + (L^2 + A)x - 2LQ - B\}^{n+\frac{1}{2}}},$$

$$(2) \quad \frac{d\psi}{dx} = 2 \frac{L - L'z}{M \sin^2 \theta} \frac{dt}{dx} \\ = \frac{(L^3 + AL - 2Q)x - 3L^2Q - 2LB + AQ}{\frac{1}{4}M^4 \sin^2 \theta \sqrt{X}},$$

$$(3) \quad p \frac{dt}{dx} = \frac{L - P}{\sqrt{X}},$$

$$(4) \quad p \frac{dt}{dx} - \frac{d\psi}{dx} \\ = \frac{-(L - P)x^2 - \{(L^3 + A)P - 2Q\}x + L^2Q + L(B + 2PQ) + BP - AQ}{\{-x^2 + (L^2 + A)x - 2LQ - B\}\sqrt{X}},$$

and thence by differentiation of (1) we can obtain sufficient relations to determine the coefficients of x in R and S , and to complete the verification.

24. These values of R and S are found to verify for the preceding case of $\mu = 6$, worked out independently.

For $\mu = 10$, $f = \frac{1}{5}$, take the expression for $I(v)$ given in L. M. S., vol. 27, p. 601, and put $\sigma - s = x$; then

$$(1) \quad x_2 = (c+1)^3(c-1)(-c^2+4c+1),$$

$$(2) \quad x_3 + x_1 = 8(c^4 + c^3 - 3c^2 - c),$$

$$(3) \quad x_1 x_3 = -16c^2(c+1)^3(c-1)(-c^2+4c+1),$$

$$(4) \quad x_3 - x_1 = 16c^2 \sqrt{(c^3 + c^2 - c)},$$

$$(5) \quad \begin{aligned} A &= -c^6 + 2c^5 + 17c^4 + 12c^3 - 31c^2 - 14c - 1, \\ B &= 8c(c+1)^3(c-1)(-c^2+4c+1)(c^3+c^2-5c-1), \end{aligned}$$

$$(6) \quad P = \frac{1}{5}(c+3)(-c^2+4c+1),$$

$$(7) \quad Q = 4c(c+1)^3(c-1)(-c^2+4c+1),$$

$$(8) \quad P_1 = (c+3)(-c^2+4c+1),$$

$$(9) \quad P_2 = 4(c+1)^3(-c^2+4c+1)^2,$$

$$(10) \quad Q_1 = 4(2c^2+5c+1)(-c^2+4c+1),$$

$$(11) \quad Q_2 = 16c(c+1)^3(-c^2+4c+1)^2,$$

$$\begin{aligned} (12) \quad I &= \int_{x_2}^x \frac{-Px + Q}{x} \frac{dx}{\sqrt{X}} \\ &= \frac{1}{5} \cos^{-1} \frac{P_1 x + P_2}{x^{\frac{3}{2}}} \sqrt{(x_3 - x)(x - x_1)} \\ &= \frac{1}{5} \sin^{-1} \frac{x^2 + Q_1 x + Q_2}{x^{\frac{5}{2}}} \sqrt{(x - x_2)}. \end{aligned}$$

Then for the curve of H ,

$$(13) \quad \left(M \frac{\rho}{h}\right)^5 e^{\gamma(\rho t - \omega)i} = E \sqrt{(x_3 - x)(x - x_1)} + iF \sqrt{(x - x_2)},$$

$$(14) \quad E = (c+3)(-c^2+4c+1)x + 4(c+1)^3(-c^2+4c+1)^2,$$

$$(15) \quad F = x^2 + 4(-c^2+4c+1)(2c^2+5c+1)x + 16c(c+1)^3(-c^2+4c+1)^2,$$

$$(16) \quad M \frac{\rho}{h} = \sqrt{x}, \quad \frac{P}{n} = \frac{L - \frac{1}{5}(c+3)(-c^2+4c+1)}{M}.$$

For the associated motion of the top,

$$(17) \quad (\tfrac{1}{2}M^2\sin\theta)^5 e^{5(pt-\psi)t} = R\sqrt{(x_3-x)(x-x_1)} + iS\sqrt{(x-x_2)},$$

where, arranged in powers of L ,

$$(18) \quad R = L^5E - 5L^4E_1 + 10L^3E_2 - 10L^2E_3 + 5LE_4 - E_5,$$

$$(19) \quad S = L^5F - 5L^4F_1 + 10L^3F_2 - 10L^2F_3 + 5LF_4 - F_5,$$

and, employing detached coefficients for the powers of c , for economy of notation,

$$(20) \quad E_1 = x^2 + (-c^2 + 4c + 1)(-c^4 - 2 + 8 + 22 + 5)x \\ - 4(c + 1)^3(-c^2 + 4 + 1)^2(c^3 + 1 - 5 - 1),$$

$$(21) \quad F_1 = P_1x^2 + 4(-c^2 + 4 + 1)(-2c^5 - 5 + 10 + 32 + 12 + 1)x \\ - 16c(c + 1)^3(-c^2 + 4 + 1)^2(c^3 + 1 - 5 - 1);$$

$$(22) \quad E_2 = P_1x^3 + (-c^2 + 4 + 1)(c^3 + 1 - 5 - 1)(c^4 + 0 - 14 - 28 - 7)x \\ + P_2(c^3 + 1 - 5 - 1)^2,$$

$$(23) \quad F_2 = x^3 + (c^6 - 2 - 25 + 0 + 115 + 50 + 5)x^2 \\ + 4(-c^2 + 4 + 1)(c^3 + 1 - 5 - 1)(2c^5 + 3 - 16 - 38 - 14 - 1)x \\ + Q_2(c^3 + 1 - 5 - 1)^2;$$

$$(24) \quad E_3 = x^3 + 2(c^6 - 2 - 17 - 2 + 61 + 28 + 3)x^2 \\ + (-c^2 + 4 + 1)(c^3 + 1 - 5 - 1)^2(-c^4 + 2 + 20 + 34 + 9)x \\ - P_2(c^3 + 1 - 5 - 1)^3,$$

$$(25) \quad F_3 = P_1x^3 + (-c^2 + 4 + 1)(c^7 + 1 - 23 - 67 + 35 + 219 + 83 + 7)x \\ + 4(-c^2 + 4 + 1)(c^3 + 1 - 5 - 1)^2(-2c^5 - 1 + 22 + 44 + 16 + 1)x \\ - Q_2(c^3 + 1 - 5 - 1)^3;$$

$$(26) \quad E_4 = P_1x^3 + 2(-c^2 + 4 + 1)(c^7 + 1 - 17 - 45 + 15 + 119 + 49 + 5)x^2 \\ + (-c^2 + 4 + 1)(c^{13} - 1 - 34 - 54 + 363 + 1037 - 588 \\ - 4068 - 1881 + 4489 + 4238 + 1402 + 205 + 11)x \\ + P_2(c^3 + 1 - 5 - 1)^4,$$

$$(27) \quad F_4 = x^4 + 2(c^6 - 2 - 21 - 6 + 73 + 32 + 3)x^3 \\ + (c^{13} - 4 - 46 + 68 + 743 + 264 - 3508 - 2952 + 5103 \\ + 4988 + 1538 + 196 + 9)x^2 \\ + (-c^2 + 4 + 1)(2c^{14} + 5 - 71 - 234 + 300 + 1719 \\ + 271 - 4732 - 3470 + 4227 \\ + 5143 + 2054 + 384 + 33 + 1)x \\ + Q^2(c^3 + 1 - 5 - 1)^4;$$

$$\begin{aligned}
 (28) \quad E_5 &= x^4 + (3c^6 - 6 - 51 - 16 + 153 + 70 + 7)x^3 \\
 &\quad + (3c^{12} - 12 - 94 + 156 + 1241 + 408 - 4676 - 4152 \\
 &\quad \quad \quad + 5365 + 5876 + 1986 + 284 + 15)x^2 \\
 &\quad + (-c^2 + 4 + 1)(-c^{16} + 2 + 54 + 54 - 946 - 2590 \\
 &\quad \quad \quad + 2214 + 13798 + 4564 - 29290 - 26254 \\
 &\quad \quad \quad + 17842 + 28754 + 13190 + 2866 + 306 + 13)x \\
 &\quad - P_2[(c^3 + 1 - 5 - 1)^5 - 32c^3(c + 1)^6(c - 1)^4]; \\
 (29) \quad F_5 &= P_1x^4 + 2(-c^2 + 4 + 1)(c^7 + 1 - 19 - 57 + 15 + 155 + 59 + 5)x^3 \\
 &\quad + (-c^2 + 4 + 1)(c^{13} - 1 - 42 - 86 + 483 + 1709 - 636 \\
 &\quad \quad \quad - 7364 - 3849 + 8937 + 7622 + 2170 + 261 + 11)x^2 \\
 &\quad + 4(-c^2 + 4 + 1)(-2c^{17} - 5 + 58 + 238 - 378 - 2554 \\
 &\quad \quad \quad + 602 + 13790 + 8130 - 30236 - 33282 \\
 &\quad \quad \quad + 15962 + 33682 + 17562 + 4462 + 602 + 40 + 1)x \\
 &\quad - Q_2[(c^3 + 1 - 5 - 1)^5 + 32c^3(c + 1)^6(c - 1)^4].
 \end{aligned}$$

Arranged in powers of x ,

$$(30) \quad R = R_0x^4 + R_1x^3 + R_2x^2 + R_3x + R_4,$$

$$(31) \quad S = S_0x^4 + S_1x^3 + S_2x^2 + S_3x + S_4,$$

$$(32) \quad R_0 = -1, \quad S_0 = 5L - P_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\begin{aligned}
 (33) \quad R_4 &= P_1 [(L + c^3 + c^2 - 5c - 1)^5 - 32c^3(c + 1)^6(c - 1)^4], \\
 S_4 &= Q_2 [(L + c^3 + c^2 - 5c - 1)^5 + 32c^3(c + 1)^6(c - 1)^4].
 \end{aligned}$$

In the upper rosette

$$(34) \quad L = \frac{Q}{x_2} = 4c, \quad M = (c - 1)^2(c + 1).$$

There are cusps on the circle $z = z_2$ when

$$(35) \quad (L - 4c)^2 = (c - 1)^4(c + 1)^2,$$

$$(36) \quad L = -c^3 + c^2 + 5c - 1 \quad \text{or} \quad c^3 - c^2 + 3c + 1;$$

and if the secular term pt is cancelled by putting

$$(37) \quad L = \frac{1}{2}(c + 3)(-c^2 + 4c + 1),$$

$$(38) \quad (c - 2)(c^2 + c - 1) = 0, \quad \text{or} \quad (3c - 1)^3 + 10 = 0.$$

With

$$(39) \quad c = 2, L = 5, p = 0, M^2 = 24\sqrt{10},$$

the motion of the top is given by

$$(40) \quad \left(\frac{1}{2}M^2\sin\theta e^{\psi i}\right)^5 = (-x^4 + 2^4 \cdot 5 \cdot 7x^3 - 2^4 \cdot 3^4 \cdot 5 \cdot 19x^2 + 2^7 \cdot 3^5 \cdot 5^2x \\ - 2^7 \cdot 3^8 \cdot 5^2 \cdot 23)\sqrt{(-x^2 + 80x + 8640)} \\ + i \cdot 2^6 \cdot 5^2(x - 135)(x^2 - 2 \cdot 3^3 \cdot 5x + 2^4 \cdot 3^5 \cdot 5)\sqrt{(x - 135)},$$

having 10 cusps at interval $\frac{1}{3}\pi$.

The other value $c = \frac{1}{3}(1 - \sqrt[3]{10})$ gives 10 cusps at interval $\frac{2}{3}\pi$; these curves are shown stereoscopically in L. M. S., vol. 27, p. 603.

In the lower rosette or intermediate curve

$$(41) \quad L = \frac{Q}{x_3 \text{ or } x_1} = -c^3 - c^2 + 3c + 1 \pm 2c\sqrt{(c^3 + c^2 - c)},$$

and with cusps on $z = z_1$,

$$(42) \quad \left(L - \frac{Q}{x_1}\right)^2 = 4\sqrt{(c^3 + c^2 - c)}[c^2 + \sqrt{(c^3 + c^2 - c)}]^2.$$

25. The degree of the results for $I(v)$ can be halved by the substitution

$$(1) \quad y^2 = \frac{x}{x - x_2},$$

and thereby the expression for the curve of H is simplified: but no reduction of degree is effected in the curve of P , because the functions of $\frac{1}{2}\chi$ in §11 are not simplified by this substitution.

Changing the limits in (2) §20,

$$(2) \quad I(v) = \int_x^{x_1} \frac{-P(v)x + Q(v)}{x} \frac{dx}{\sqrt{X}} \\ = \frac{1}{2n+1} \cos^{-1} \frac{x^n + Q_1x^{n-1} + \dots + Q_n}{x^{n+1}} \sqrt{(x - x_2)} \\ = \frac{1}{2n+1} \cos^{-1} \left(1 + \frac{Q_1}{x} + \dots + \frac{Q_n}{x^n}\right) \sqrt{\left(1 - \frac{x_2}{x}\right)} \\ = \frac{1}{2n+1} \cos^{-1} \left\{1 + \frac{Q_1}{x_2} \left(\frac{y^2 - 1}{y^2}\right) + \dots + \frac{Q_n}{x_2^n} \left(\frac{y^2 - 1}{y^2}\right)^n\right\} \frac{1}{y} \\ = \frac{1}{2n+1} \cos^{-1} \frac{R_0y^{2n} + R_1y^{2n-2} + \dots}{y^{2n+1}}$$

$$\begin{aligned}
&= \frac{2}{2n+1} \cos^{-1} \frac{y^{n-1} + A_1 y^{n-2} + \dots + A_{n-1}}{y^{n+\frac{1}{2}}} \\
&\quad \times \sqrt{\frac{1}{2}(y+1)(y^2 + C_1 y + C_2)} \} \\
&= \frac{2}{2n+1} \sin^{-1} \frac{y^{n-1} - A_1 y^{n-2} + \dots + (-1)^{n-1} A_{n-1}}{y^{n+\frac{1}{2}}} \\
&\quad \times \sqrt{\frac{1}{2}(y-1)(y^2 - C_1 y + C_2)} \},
\end{aligned}$$

where the coefficients $A_1, \dots, A_{n-1}$ are less than a quarter in number of the $P_1, \dots, P_n, Q_1, \dots, Q_n$, required previously.

Changing the variable from x to y in the integral $I(v)$, with the notation

$$(3) \quad y_1^2 = \frac{x_1}{x_1 - x_2} = \operatorname{sn}^2 fK',$$

$$(4) \quad y_3^2 = \frac{x_3}{x_3 - x_2} = \frac{1}{\operatorname{dn}^2(1-f)K'},$$

$$\begin{aligned}
(5) \quad Y &= 4(y^2 - 1)(y^2 - y_1^2)(y^2 - y_3^2) \\
&= \frac{x_2^3}{(x_3 - x_2)(x_2 - x_1)} \cdot \frac{X}{(x - x_2)^4},
\end{aligned}$$

$$(6) \quad \int_{y_3}^y \frac{dy^2}{\sqrt{Y}} = \sqrt{\frac{(x_3 - x_2)(x_2 - x_1)}{x_2}} \int_x^{x_3} \frac{dx}{\sqrt{X}} = \frac{\sqrt{\frac{-x_1 x_3}{x_2}}}{y_1 y_3} \int \frac{dx}{\sqrt{X}},$$

$$\begin{aligned}
(7) \quad I(v) &= \int_{y_3}^y \frac{\left(\sqrt{\frac{-x_1 x_3}{x_2}} - P\right)y^2 - \sqrt{\frac{-x_1 x_3}{x_2}}}{y^2} \cdot \frac{y_1 y_3}{\sqrt{\frac{-x_1 x_3}{x_2}}} \frac{dy^2}{\sqrt{Y}} \\
&= \int \frac{y_1 y_3 \sqrt{\frac{x_2}{-x_1 x_3}} P(\omega_2 - v) y^2 - y_1 y_3}{y^2} \frac{dy^2}{\sqrt{Y}} \\
&= \int \frac{P'(v) y^2 - Q'(v)}{y^2} \frac{dy^2}{\sqrt{Y}}
\end{aligned}$$

$$(8) \quad P'(v) = y_1 y_3 \sqrt{\frac{x_2}{-x_1 x_3}} \cdot P(\omega_2 - v), \quad Q'(v) = y_1 y_3 = \frac{\operatorname{sn} fK'}{\operatorname{dn}(1-f)K'}.$$

Also we may take

$$(9) \quad y^2 \pm C_1 y + C_2 = (y \pm y_1)(y \mp y_3)$$

$$(10) \quad x_3 > x > x_2 > 0 > x_1,$$

$$(11) \quad \infty > y^2 > y_3^2 > 1 > y_1^2,$$

$$(12) \quad \frac{P(v)}{\sqrt{(y_3^2 - y_1)}} = \frac{P(\omega_2 - v)}{\sqrt{(x_3 - x_1)}} = z s(1 - f') K' = - z c f' K'.$$

26. Thus for $\mu = 6$, with this reduction,

$$(1) \quad \begin{aligned} I(v) &= \frac{2}{3} \cos^{-1} \frac{\sqrt{\frac{1}{2}(y+1)(y^2 + ay - a)}}{y^{\frac{3}{2}}} \\ &= \frac{2}{3} \sin^{-1} \frac{\sqrt{\frac{1}{2}(y-1)(y^2 - ay - a)}}{y^{\frac{3}{2}}} \\ &= \int \frac{\frac{1}{2}(1+a)y^2 - a}{y^2} \frac{dy^2}{\sqrt{Y}}. \end{aligned}$$

Similarly, for $\mu = 10$,

$$(2) \quad \begin{aligned} I(v) &= \frac{2}{5} \cos^{-1} \frac{y - \frac{2}{c+1}}{y^{\frac{5}{2}}} \sqrt{\left[\frac{1}{2}(y+1) \left\{ y^2 + \frac{4c^2y + 4c}{(c+1)(c-1)^2} \right\} \right]} \\ &= \frac{2}{5} \sin^{-1} \frac{y + \frac{2}{c+1}}{y^{\frac{5}{2}}} \sqrt{\left[\frac{1}{2}(y-1) \left\{ y^2 + \frac{-4c^2y + 4c}{(c+1)(c-1)^2} \right\} \right]} \\ &= \int \frac{\frac{c^3 - c^2 + 7c - 3}{5(c+1)(c-1)^2} y^2 - \frac{4c}{(c+1)(c-1)^2}}{y^2} \frac{dy^2}{\sqrt{Y}}. \end{aligned}$$

The next case of $\mu = 14$ can now be attempted; utilising the analysis in L. M. S., vol. 25, p. 257, and vol. 27, p. 462, we can put

$$(3) \quad C_1 = \frac{(1-2m)(2-3m-\sqrt{M})}{4m(1-m)^3}, \quad C_2 = \frac{(1-2m)^2(2-3m-\sqrt{M})}{4m(1-m)^3},$$

where

$$(4) \quad M = 4m - 11m^2 + 8m^3,$$

$$(5) \quad A_1 = - \frac{[\sqrt{(4-11m+8m^2)} - \sqrt{m}]^2}{4m(1-m)},$$

$$A_2 = \frac{(1-2m)[\sqrt{(4-11m+8m^2)} - \sqrt{m}]^2}{4m(1-m)^2},$$

$$7P'(v) = \frac{-6 + 33m - 70m^2 + 64m^3 - 20m^4 + (3 - 6m + 4m^2)\sqrt{M}}{4m(1-m)^3},$$

where m is an arbitrary parameter.

The herpolhode curve of H can now be written down ; but the associated motion of the top will be of very complicated character.

The result of $I(v)$ for $\mu = 18$ has been given in the *Archiv der Mathematik und Physik*, ser. 3, vol. 1, p. 73 ; and the preliminaries for $\mu = 22$ have been sketched out in the *Bulletin de la Société mathématique de France*, vol. 29, 1901, requisite for the calculation of C_1 , C_2 , $P'(v)$, A_1 , A_2 , A_3 , A_4 , in terms of a single parameter.

But the next cases of $\mu = 26$ and $\mu = 30$ still await solution.

27. With $\mu = 4n$, a further reduction is possible which makes the expression for the path of P or C' assume a symmetrical form, as was seen in the case discussed already, of

$$(1) \quad \mu = 4, \quad \text{and} \quad v = K + \frac{1}{2}K'.$$

Thus with $\mu = 8$, and

$$(2) \quad v = K + fK', \quad f = \frac{1}{4} \text{ or } \frac{3}{4},$$

we shall find that, putting in the standard form of $I(v)$ in (1) §10,

$$(3) \quad s - s_3 = x^2, \quad \sigma - s_3 = \delta, \quad \sigma - s = \delta - x^2,$$

$$\begin{aligned} (4) \quad I(v) &= \int_x^k \frac{P(v)(\delta - x^2) + Q(v)}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}} \\ &= \frac{1}{2} \cos^{-1} \frac{\frac{1}{2}(1+a)^2 - x}{\delta - x^2} \sqrt{\frac{1}{2}X_1} \\ &= \frac{1}{2} \sin^{-1} \frac{\frac{1}{2}(1+a)^2 + x}{\delta - x^2} \sqrt{\frac{1}{2}X_2}, \end{aligned}$$

where

$$(5) \quad X_1 = (1+x) \left\{ \frac{1}{4} \left(\frac{1}{a} - a \right) + x \right\} = (1+x)(k+x),$$

$$X_2 = (1-x) \left\{ \frac{1}{4} \left(\frac{1}{a} - a \right)^2 - x \right\} = (1-x)(k-x),$$

$$(6) \quad k = \frac{1}{4} \left(\frac{1}{a} - a \right)^2, \quad x = k \operatorname{sn}(K - mt);$$

as for $\mu = 4$; and the octahedron irrationality

$$(7) \quad o = \sqrt{k} = \frac{1}{2} \left(\frac{1}{a} - a \right),$$

unchanged by the substitution $\left(a, -\frac{1}{a}\right)$, which interchanges $f = \frac{1}{4}$ and $\frac{3}{4}$.

Also

$$(8) \quad \delta = \frac{(1+a)^3(1-a)}{4a},$$

$$(9) \quad P(v) = \frac{(1+2a+3a^2)(-1+2a+a^2)}{16a^2},$$

$$(10) \quad Q(v) = \frac{(1+a)^2(1-a^4)(-1+2a+a^2)}{16a^3},$$

$$(11) \quad \operatorname{dn}^2 fH' = \frac{\sigma - s_3}{s_1 - s_3} = \delta,$$

$$\operatorname{sn}^2 fH' = \frac{4a^3}{(1+a^2)(1+2a-a^2)}, \quad \operatorname{cn}^2 fH' = \frac{(1+a)^3(1-a)}{(1+a^2)(1+2a-a^2)},$$

$$(12) \quad \operatorname{cn} 2fH' = \frac{1-a^2}{1+a^2}, \quad \operatorname{dn} 2fH' = o = \frac{1}{2} \left(\frac{1}{a} - a \right).$$

The vector of H is given by

$$(13) \quad M \frac{\rho}{c} e^{(\mu t - \omega)i} = \left[\frac{1}{2}(1+a)^2 - x \left\{ \sqrt{\frac{1}{2}X_1} + i \left\{ \frac{1}{2}(1+a)^2 + x \left\{ \sqrt{\frac{1}{2}X_2} \right\} \right\} \right\} \right],$$

$$(14) \quad \frac{1}{2} M^2 \sin \theta e^{xi} = \frac{L(\delta - x^2) - Q + ix \sqrt{X_1 X_2}}{\sqrt{(\delta - x^2)}},$$

$$(15) \quad \left(\frac{1}{2} M^2 \sin \theta \right)^2 e^{2(\mu t - \psi)i}$$

$$= \frac{[L(\delta - x^2) - Q + ix \sqrt{X_1 X_2}]^2}{(\delta - x^2)^2}$$

$$\times \left[\frac{1}{2}(1+a)^2 - x \left\{ \sqrt{\frac{1}{2}X_1} + i \left\{ \frac{1}{2}(1+a)^2 + x \left\{ \sqrt{\frac{1}{2}X_2} \right\} \right\} \right\} \right]$$

$$= (A_0 + A_1 x + A_2 x^2 + x^3) \sqrt{\frac{1}{2}X_1} + i (A_0 - A_1 x + A_2 x^2 - x^3) \sqrt{\frac{1}{2}X_2},$$

since the factor $(\delta - x^2)^2$ cancels; and we find

$$(16) \quad A_0 = \frac{1}{2}(1 + a)^2 \left\{ L - \frac{(1 + a^2)(-1 + 2a + a^2)}{4a^2} \right\},$$

$$(17) \quad A_1 = - \left\{ L - \frac{(1 + a^2)(-1 + 2a + a^2)}{4a^2} \right\} \left\{ L + \frac{1 + 2a + 0 - 6a^3 - a^4}{4a^2} \right\},$$

$$(18) \quad A_2 = 2L - \frac{1}{2}(1 + a)^2.$$

In a lower rosette

$$(19) \quad L = \frac{Q}{\delta} = \frac{(1 + a^2)(-1 + 2a + a^2)}{4a^2}, \quad A_0 = 0, \quad A_1 = 0, \quad A_2 = -\frac{1}{2} \left(\frac{1 - a}{a} \right)^2$$

$$(20) \quad M \cos \frac{1}{2}\theta = x, \quad M^2 = \frac{(1 + a)(1 - a)^3}{4a^2},$$

$$(21) \quad \tan^2 \frac{1}{2}\theta e^{2(\rho t - \psi)i} = \frac{(A_2 + x)\sqrt{\frac{1}{2}X_1} + i(A_1 - x)\sqrt{\frac{1}{2}X_2}}{x^2}.$$

In an upper rosette

$$(22) \quad L = \frac{Q}{\delta - k^2} = a, \quad M^2 = \frac{(1 - a)^2(1 - a^4)(1 + 2a - a^2)}{16a^4},$$

$$M^2 \sin^2 \frac{1}{2}\theta = k^2 - x^2,$$

$$(23) \quad \cot^2 \frac{1}{2}\theta e^{2(\rho t - \psi)i} = \frac{\frac{1}{2}(1 - a)^2 - x}{k - x} \sqrt{\left(\frac{1}{2} \frac{1 + x}{k + x}\right)} \\ + i \frac{\frac{1}{2}(1 - a)^2 + x}{k + x} \sqrt{\left(\frac{1}{2} \frac{1 - x}{k - x}\right)}.$$

In the intermediate case

$$(24) \quad L = \frac{Q}{\delta - 1} = -\frac{(1 + a)^3(1 - a)}{4a^2}, \quad M^2 = \frac{(1 + a^2)(1 + 2a - a^2)}{4a^3},$$

$$M^2 \sin^2 \frac{1}{2}\theta = 1 - x^2,$$

$$(25) \quad \cot^2 \frac{1}{2}\theta e^{2(\rho t - \psi)i} = \frac{\frac{1}{2} \left(\frac{1 + a}{a} \right)^2 - x}{1 - x} \sqrt{\left(\frac{1}{2} \frac{k + x}{1 + x} \right)} \\ + i \frac{\frac{1}{2} \left(\frac{1 + a}{a} \right)^2 + x}{1 + x} \sqrt{\left(\frac{1}{2} \frac{k - x}{1 - x} \right)}.$$

There are cusps on the circle $z = z_2$ when

$$(26) \quad (L - a)^2 = \frac{(1 - a)^2(1 - a^4)(1 + 2a - a^2)}{16a^4};$$

and in the algebraical case when

$$(27) \quad L = P, \quad L - a = -\frac{(1 + 2a - a^2)(1 - 2a + 3a^2)}{16a^2},$$

$$(28) \quad (1 + 2a - a^2)(1 - 2a + 3a^2)^2 - 16(1 - a)^2(1 - a^4) = 0,$$

$$(29) \quad 7a^6 - 2a^5 - 9a^4 + 12a^3 - 15a^2 + 30a - 15 = 0,$$

(L. M. S., vol. 27, p. 180), a sextic equation having two real roots

$$(30) \quad a = 0.642, \text{ and } -1.812, \text{ giving 8 cusps at interval } \frac{1}{4}\pi, \text{ and } \frac{3}{4}\pi,$$

as shown in the stereoscopic figure in L. M. S., vol. 27, p. 599.

28. We can now complete the case left unfinished in L. M. S., vol. 27, p. 605, of twelve branched gyroscope curves, given by

$$(1) \quad \mu = 12, \quad v = K + fK'i, \quad f = \frac{1, 5}{6};$$

$$(2) \quad I = \int_x^{x_2} \frac{P(\delta - x^2) + Q}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}}$$

$$= \frac{1}{3} \cos^{-1} \frac{1 + a + a^2 + (1 - a)x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\frac{1}{2} X_1}$$

$$= \frac{1}{3} \sin^{-1} \frac{1 + a + a^2 - (1 - a)x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\frac{1}{2} X_2},$$

$$(3) \quad X_1, X_2 = a^3(1 + a + a^2) \pm \frac{1}{2}(1 + a)^3(1 - a)x - x^2,$$

$$(4) \quad \delta = a + a^2 + a^3,$$

$$(5) \quad P = \frac{1}{12}(1 - a)(5 + 3a + 3a^2 + a^3),$$

$$(6) \quad Q = \frac{1}{2}(1 - a^4)(a + a^2 + a^3);$$

$$(7) \quad \left(M \frac{\rho}{c}\right)^3 e^{3(pt-\omega)i} = \{1 + a + a^2 + (1-a)x - x^2\} \sqrt{\frac{1}{2}X_1} \\ + i \{1 + a + a^2 - (1-a)x - x^2\} \sqrt{\frac{1}{2}X_2},$$

$$(8) \quad \frac{\rho^2}{c^2} = \frac{a + a^2 + a^3 - x^2}{M^2},$$

$$(9) \quad x = x_2 \sin(K - mt).$$

For the curve of P , working out the product of $(\frac{1}{2} M^2 \sin \theta e^{xi})^3$ and e^{3ti} , the factor $(\delta - x^2)^3$ cancels, and

$$(10) \quad (\tfrac{1}{2} M^2 \sin \theta)^3 e^{3(pt-\psi)i} = (A_0 - A_1x + \dots + A_4x^4 - x^5) \sqrt{\tfrac{1}{2}X_1} \\ + i (A_0 + A_1x + \dots + A_4x^4 + x^5) \sqrt{\tfrac{1}{2}X_2}.$$

I am indebted to Col. C. E. Basevi for the calculation of these A coefficients in this and the preceding case for $\mu = 8$; he finds

$$(11) \quad A_0 = (1 + a + a^2)L - \tfrac{1}{2}(1 - a^4)\}^3, \\ A_1 = -\{L - \tfrac{1}{2}(1 - a^4)\}^2\{(1 - a)L - \tfrac{1}{2}(1 - a + 6a^2 + 6a^3 + 5a^4 + a^5)\}, \\ A_2 = -\{L - \tfrac{1}{2}(1 - a^4)\}\{L^2 - (1 - a)(4 + 4a + 4a^2 - a^3)L \\ + \tfrac{1}{4}(7 + 0 + 0 - 6a^3 + 4a^4 + 12a^5 + 12a^6 + 6a^7 + a^8)\}, \\ A_3 = -3L^2 + 3(1 - a)L - \tfrac{1}{4}(3 - 6a + 0 + 0 + 0 + 6a^5 + 4a^6 + 4a^7 + a^8), \\ A_4 = 3L - \tfrac{1}{2}(1 - a)(5 + 3a + 3a^2 + a^3).$$

In the lower rosette

$$(12) \quad L = \tfrac{1}{2}(1 - a^4), \quad A_0 = A_1 = A_2 = 0, \\ A_3 = -a^6(1 + a + a^2), \quad A_4 = (1 - a)(1 + a^3), \quad M \cos \tfrac{1}{2} \theta = x, \\ (13) \quad M^3 \sin^3 \tfrac{1}{2} \theta e^{3(pt-\psi)i} = x^3 \tan^3 \tfrac{1}{2} \theta e^{3(pt-\psi)i} \\ = (-A_3 + A_4x - x^2) \sqrt{\tfrac{1}{2}X_1} + i(A_3 + A_4x + x^2) \sqrt{\tfrac{1}{2}X_2}.$$

For the upper rosettes and intermediate path the condition

$$(14) \quad \left(L - \frac{Q}{\delta - x_1^2}\right) \left(L - \frac{Q}{\delta - x_2^2}\right) = 0$$

reduces to

$$(15) \quad L^2 - 1(1 - 4a - 4a^2 - 4a^3 - a^4)L - a(1 + a + a^2) = 0, \\ L = \frac{1}{4} \{ 1 - 4a - 4a^2 - 4a^3 - a^4 \pm (1 + a)^2 \sqrt{(1 + a^2)(1 + 4a + a^2)} \}.$$

In fact

$$(16) \quad x_1, x_2 = \pm \frac{1}{4}(1 + a)^3(1 - a) + \frac{1}{4}(1 + a^2)\sqrt{(1 + a^2)(1 + 4a + a^2)},$$

$$(17) \quad o = \sqrt{k} = \sqrt{\frac{x_2}{x_1}} = \frac{-(1 + a)^3(1 - a) + (1 + a^2)\sqrt{A}}{4\sqrt{(a^3 + a^4 + a^5)}},$$

$$A = (1 + a^2)(1 + 4a + a^2).$$

$$(18) \quad \left(\frac{1}{o} - o\right)^2 = \frac{(1 + a)^6(1 - a)^2}{4a^3(1 + a + a^2)} = \frac{(b + 1)^3(b - 3)}{4b},$$

putting $a + 1 + \frac{1}{a} = b :$

$$(19) \quad \left(\frac{1}{o} + o\right)^2 = \frac{(b - 1)^3(b + 3)}{4b},$$

$$(20) \quad \lambda^2 = \frac{4}{\left(\frac{1}{o} + o\right)^2} = \frac{16b}{(b - 1)^3(b + 3)}, \quad \lambda'^2 = \left(\frac{b + 1}{b - 1}\right)^3 \frac{b - 3}{b + 3}.$$

$$(21) \quad \text{dn}^2 = fK' \frac{\delta}{x_1^2} = \frac{x_2^2}{a^4(1 + a + a^2)}, \quad \frac{\text{dn} fK'}{o} = \frac{1}{a}.$$

The cusp condition is

$$(22) \quad \left(L - \frac{Q}{\delta - x_2^2}\right)^2 = \frac{(x_1^2 - x_2^2)(x_2^2)}{\delta - x_2^2};$$

and becomes when rationalised in the algebraical case, when $L = P$, an equation of the 14th degree :

$$(23) \quad 875a^{14} + 2660 + 2765 + 280 - 4025 - 8900 - 10815 - 8930 \\ - 4767 - 1124 + 295 + 280 + 173 + 68 + 11 = 0.$$

29. We are now prepared to generalise for μ a multiple of 4, and first we notice that we must distinguish the two cases of

$$(1) \quad \mu = 8n, \text{ or } 8n + 4,$$

$$(2) \quad v = H' + \frac{rK'i}{4n}, \text{ or } K + \frac{rK'i}{4n+2};$$

$$(3) \quad I = \int_x^{x_2} \frac{-P(\delta - x^2) + Q}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}},$$

and according as $\frac{1}{2}\mu = 2n$, or $\frac{1}{2}\mu = 2n + 1$,

$$(4) \quad \begin{aligned} X_1 &= P_0 + P_1 x + x^2, & \text{or} & & P_0 + P_1 x - x^2, \\ &= (x_1 + x)(x_2 + x) & \text{or} & & (x_1 - x)(x_2 + x), \end{aligned}$$

$$(5) \quad \begin{aligned} X_2 &= P_0 - P_1 x + x^2, & \text{or} & & P_0 - P_1 x - x^2, \\ &= (x_1 - x)(x_2 - x), & \text{or} & & (x_1 + x)(x_2 - x), \end{aligned}$$

$$(6) \quad x_2 = kx_1, \quad \text{or} \quad x_1 = \frac{1}{o}, \quad x_2 = o,$$

when normalised so as to make $P_0 = 1$;

$$(7) \quad \delta = s(v) - s_3 = x_1^2 \operatorname{dn}^2 fH',$$

$$(8) \quad P_0 = s_3 - s(2nv), \quad \text{or} \quad P_0 = s(2n+1)v - s_3,$$

$$(9) \quad P_1 = 2P(2nv), \quad \text{or} \quad P_1 = 2P(2n+1)v,$$

$$(10) \quad \left(\frac{1}{o} + o\right)^2 = \frac{P_1^2}{P_0}, \quad \text{or} \quad \left(\frac{1}{o} - o\right)^2 = \frac{P_1^2}{P_0},$$

$$(11) \quad x = x_2 \operatorname{sn}(H' - mt), \quad mt = \int_x^{x_2} \frac{x_1 dx}{\sqrt{X_1 X_2}}.$$

$$(12) \quad I = \frac{1}{2n} \cos^{-1} \frac{R_0 + R_1 x + \dots + R_{2n-1} x^{2n-1}}{(\delta - x^2)^n} \sqrt{\frac{1}{2} X_1},$$

$$\begin{aligned} \text{or } & \frac{1}{2n+1} \cos^{-1} \frac{R_0 + R_1 x + \dots + R_{2n} x^{2n}}{(\delta - x^2)^{n+\frac{1}{2}}} \sqrt{\frac{1}{2} X_1} \\ &= \frac{1}{2n} \sin^{-1} \frac{R_0 - R_1 x + \dots - R_{2n-1} x^{2n-1}}{(\delta - x^2)^n} \sqrt{\frac{1}{2} X_2}, \end{aligned}$$

$$\text{or } \frac{1}{2n+1} \sin^{-1} \frac{R_0 - R_1 x + \dots + R_{2n} x^{2n}}{(\delta - x^2)^{n+\frac{1}{2}}} \sqrt{\frac{1}{2} X_2},$$

$$(13) \quad R_{2n-1} = (-1)^n, \quad R_{2n} = (-1)^n.$$

The vector of H is then given by

$$\begin{aligned}
 (14) \quad M \frac{\rho}{c} e^{(pt - \omega)t} &= \sqrt{(\delta - x^2)} e^{it} \\
 &= \{ (R_0 + R_1x + \dots + R_{2n-1}x^{2n-1}) \sqrt{\frac{1}{2} \bar{X}_1} \\
 &\quad + i(R_0 - R_1x + \dots - R_{2n-1}x^{2n-1}) \sqrt{\frac{1}{2} \bar{X}_2} \}^{\frac{1}{2n}}, \\
 &\text{or } \{ (R_0 + R_1x + \dots + R_{2n}x^{2n}) \sqrt{\frac{1}{2} \bar{X}_1} \\
 &\quad + i(R_0 - R_1x + \dots + R_{2n}x^{2n}) \sqrt{\frac{1}{2} \bar{X}_2} \}^{\frac{1}{2n+1}}
 \end{aligned}$$

and multiplying $\frac{1}{2} M^2$ in θe^{xt} by e^{it} , rationalising, and cancelling the factor

$$(\delta - x^2)^{2n, \text{ or } 2n+1},$$

$$\begin{aligned}
 (15) \quad \left\{ \frac{1}{2} M^2 \sin \theta e^{(pt - \psi)t} \right\}^{2n, \text{ or } 2n+1} \\
 &= (A_0 + A_1x + \dots + A_{4n-1}x^{4n-1}) \sqrt{\frac{1}{2} \bar{X}_1} \\
 &\quad + i(A_0 - A_1x + \dots - A_{4n-1}x^{4n-1}) \sqrt{\frac{1}{2} \bar{X}_2}, \\
 &\text{or } (A_0 + A_1x + \dots + A_{4n+1}x^{4n+1}) \sqrt{\frac{1}{2} \bar{X}_1} \\
 &\quad + i(A_0 - A_1x + \dots - A_{4n+1}x^{4n+1}) \sqrt{\frac{1}{2} \bar{X}_2};
 \end{aligned}$$

where the determination of the coefficients R and A is a piece of algebraical labour of increasing scale as μ increases.

30. Putting in the standard form of (1) §10

$$(1) \quad \sigma - s_1, \sigma - s_2, \sigma - s_3 = \sigma_1, \sigma_2, \sigma_3 \text{ (denoted previously by } x_1, x_2, x_3),$$

$$(2) \quad \sigma_1 = \delta - x_1^2, \sigma_2 = \delta - x_2^2, \sigma_3 = \delta,$$

$$(3) \quad 3\wp v = \sigma_1 + \sigma_2 + \sigma_3 = 3\delta - x_1^2 - x_2^2 = x_1^2(1 + k'^2 - 3k'^2 \operatorname{sn}^2 fK'),$$

$$\begin{aligned}
 (4) \quad -\frac{1}{2} i\wp'v = Q = \sqrt{(-\sigma_1, \sigma_2, \sigma_3)} &= \sqrt{(x_1^2 - \delta)(\delta - x_2^2)(\delta)} \\
 &= x_1^3 k'^2 \operatorname{sn} fK' \operatorname{cn} fK' \operatorname{dn} fK',
 \end{aligned}$$

$$(5) \quad \frac{1}{2} \wp''v = \sigma_2 \sigma_3 + \sigma_1 \sigma_1 + \sigma_1 \sigma_2 = 3\delta^2 - 2(x_1^2 + x_2^2) \delta + x_1^2 x_2^2,$$

$$(6) \quad M^2 z = L^2 + \delta - x_1^2 - x_2^2 + 2x^2,$$

$$(7) \quad \frac{1}{2} M^2 (E - z) = \sigma - s = \delta - x^2,$$

$$(8) \quad \frac{1}{2} M^2 (z - z_3) = s - s_3 = x^2,$$

$$(9) \quad \frac{1}{2} M^2 (z_2 - z) = s_2 - s = x_2^2 - x^2,$$

$$(10) \quad \frac{1}{2} M^2 (z_1 - z) = s_1 - s = x_1^2 - x^2.$$

From (7) §11,

$$(11) \quad \frac{1}{4} M^4 \sin^2 \theta = -2 L Q - \frac{1}{2} \varphi''(v) + (L^2 + 3\varphi v)(\sigma - s) - (\sigma - s)^2 \\ = [L \sqrt{\delta} \sqrt{(x_1^2 - \delta)(\delta - x_2^2)}]^2 - (L^2 + \delta - x_1^2 - x_2^2) x^2 - x^4,$$

$$(12) \quad M^4 = (L^2 + \delta - x_1^2 - x_2^2)^2 + 4 [L \sqrt{\delta} - \sqrt{(x_1^2 - \delta)(\delta - x_2^2)}]^2,$$

$$(13) \quad \frac{1}{2} M^2 \sin \theta_3 = L \sqrt{\delta} - \sqrt{(x_1^2 - \delta)(\delta - x_2^2)}.$$

In lower rosettes, writing f for fK' ,

$$(14) \quad L \sqrt{\delta} = \sqrt{(x_1^2 - \delta)(\delta - x_2^2)}, \quad L = x_1 k'^2 \frac{\operatorname{sn} f \operatorname{cn} f}{\operatorname{dn} f}, \quad M \cos \frac{1}{2} \theta = x,$$

$$(15) \quad M^2 \sin^2 \frac{1}{2} \theta = -L^2 - \delta + x_1^2 + x_2^2 - x^2,$$

$$(16) \quad M^2 = -L^2 - \delta + x_1^2 + x_2^2 = \frac{x_1^2 x_2^2}{\delta}, \quad M = x_1 \operatorname{dn}(1 - f) K'.$$

In upper rosettes (or intermediate path),

$$(17) \quad L \sqrt{(\delta - x_2^2)} = \sqrt{(x_1^2 - \delta)\delta}, \quad L = x_1 \frac{\operatorname{sn} f \operatorname{dn} f}{\operatorname{cn} f} \left(\text{or } x_1 \frac{\operatorname{cn} f \operatorname{dn} f}{\operatorname{sn} f} \right),$$

$$(18) \quad \frac{1}{2} M^2 (1 - z) = x_2^2 - x^2 \text{ (or } x_1^2 - x^2),$$

$$(19) \quad \frac{1}{2} M^2 (1 + z) = L^2 + \delta - x_1^2 + x^2 = \frac{x_1^2 - \delta}{\delta - x_2^2} x_2^2 + x^2,$$

$$(20) \quad M^2 = x_2^2 \frac{x_1^2 - x_2^2}{\delta - x_2^2}, \quad M = \frac{k x_1}{\operatorname{cn} f} \text{ or } \frac{x_1}{\operatorname{sn} f}.$$

For cusps on the circle $z = z_2$ (or z_1)

$$(21) \quad \left[L - \sqrt{\left(\frac{x_1^2 - \delta}{\delta - x_2^2} \delta \right)} \right]^2 = \frac{x_1^2 - x_2^2}{\delta - x_2^2} x_2^2,$$

$$(22) \quad L = \frac{Q + x_2 \sqrt{(x_1^2 - x_2^2)(\delta - x_2^2)}}{\delta - x_2^2} \\ = x_1 \frac{\operatorname{sn} f \operatorname{dn} f + k}{\operatorname{cn} f}.$$

The calculations can now be carried out for $\mu = 16, 24, 32, 40$, and $\mu = 20, 28, 36, 44$; but $\mu = 52$ and 60 must await the reduction of $\mu = 26$ and 30 , not yet accomplished.

31. The chief practical interest is satisfied by the determination of the motion of the axis of the top, by Euler's angles θ and ψ ; but to complete the solution theoretically we must determine Euler's third angle ϕ , and thence Klein's $\alpha, \beta, \gamma, \delta$.

If v_1, v_2 , denote the values of u corresponding to $z = -1, 1$, then since

$$(1) \quad z_1 > 1 > z_2 > z > z_3 > -1,$$

we must take

$$(2) \quad v_1 = f_1 \omega_3, \quad v_2 = \omega_1 + f_2 \omega_3,$$

where f_1, f_2 are real fractions; and v, w corresponding to $z = E, D$.

$$(3) \quad v = v_1 - v_2, \quad w = v_1 + v_2.$$

In the spherical top in which $A_1 = C_1$, which we may take without essential loss of generality,

$$(4) \quad \frac{d\phi}{dt} = R - \cos \theta \frac{d\psi}{dt} = \frac{G'}{A_1} - \cos \theta \frac{d\psi}{dt},$$

$$(5) \quad \frac{d\psi}{dt} = \frac{G - G' \cos \theta}{A_1 \sin^2 \theta},$$

$$(6) \quad \frac{d\phi}{dt} = \frac{G' - G \cos \theta}{A_1 \sin^2 \theta},$$

$$(7) \quad \frac{d(\phi + \psi)}{dt} = \frac{G' + G}{A_1(1 + \cos \theta)}, \quad \phi + \psi = 2 \frac{L' + L}{M} \int_z^{z_1} \frac{dz}{(1+z)\sqrt{2Z}},$$

$$(8) \quad \frac{d(\phi - \psi)}{dt} = \frac{G' - G}{A_1(1 - \cos \theta)}, \quad \phi - \psi = 2 \frac{L' - L}{M} \int \frac{dz}{(1-z)\sqrt{2Z}}.$$

Now with

$$(9) \quad \frac{1}{2} M^2 (1 + z) = M^2 \cos^2 \frac{1}{2} \theta = s - s(v_1),$$

$$(10) \quad \frac{1}{2} M^2 (1 - z) = M^2 \sin^2 \frac{1}{2} \theta = s(v_2) - s,$$

$$(11) \quad M^2 = s(v_2) - s(v_1),$$

$$(12) \quad \wp'^2 u = S = \frac{1}{2} M^6 Z,$$

$$(13) \quad -S(v_1) = M^4 (L' + L)^2, \quad -S(v_2) = M^4 (L' - L)^2.$$

Then

$$(14) \quad \frac{1}{2}(\phi + \psi) = \int_s^{s_2} \frac{\frac{1}{2}\sqrt{-S(v_1)}}{s - s(v_1)} \frac{ds}{\sqrt{S}}$$

$$(15) \quad \frac{1}{2}(\phi - \psi) = \int \frac{\frac{1}{2}\sqrt{-S(v_2)}}{s(v_2) - s} \frac{ds}{\sqrt{S}}$$

or

$$(16) \quad \frac{1}{2}(\phi + \psi) = \frac{P(v_1)}{M} nt - I(v_1) = p_1 t - I(v_1),$$

$$(17) \quad \frac{1}{2}(\phi - \psi) = \frac{P(v_2)}{M} nt + I(v_2) = p_2 t + I(v_2),$$

where

$$(18) \quad I(v_1) = \int_s^{s_2} \frac{P(v_1)[s - s(v_1)] - \frac{1}{2}\sqrt{-S(v_1)}}{s - s(v_1)} \frac{ds}{\sqrt{S}},$$

$$(19) \quad I(v_2) = \int \frac{-P(v_2)[s(v_2) - s] + \frac{1}{2}\sqrt{-S(v_2)}}{s(v_2) - s} \frac{ds}{\sqrt{S}},$$

$$(20) \quad I(v) = \frac{1}{2} i \log \frac{\theta(u + v)}{\theta(u - v)}.$$

Now, introducing Klein's functions, α , β , γ , δ , cited at the outset,

$$(21) \quad \begin{aligned} M\alpha &= \sqrt{s - s(v_1)} e^{[p_1 t - I(v_1)] i} \\ &= e^{p_1 t} \frac{\theta(0)\theta(u + v_1)}{\lambda \theta u \theta v_1} = e^{p_1 t} (A_1 + B_1 i), \end{aligned}$$

suppose; and

$$(22) \quad M\delta = e^{-p_1 t} \frac{\theta(0)\theta(u + v_1)}{\lambda \theta u \theta v_1} = e^{-p_1 t} (A_1 - B_1 i),$$

with

$$(23) \quad M^2 \alpha \delta = s - s(v_1) = A_1^2 + B_1^2.$$

Also

$$(24) \quad \begin{aligned} M\gamma &= i \sqrt{s(v_2) - s} e^{[p_2 t + I(v_2)] i} \\ &= i e^{p_2 t} \frac{\theta(0)\theta(u - v_2)}{\lambda \theta u \theta v_2} = e^{p_2 t} (i A_2 - B_2 i), \end{aligned}$$

$$(25) \quad M\beta = i e^{-p_2 t} \frac{\theta(0)\theta(u + v_2)}{\lambda \theta u \theta v_2} = e^{-p_2 t} (i A_2 + B_2 i),$$

$$(26) \quad M^2 \beta \gamma = s - s(v_2) = -A_2^2 - B_2^2,$$

$$(27) \quad \begin{aligned} M^2(a\delta - \beta\gamma) &= s(v_2) - s(v_1) \\ &= M^2 = A_1^2 + B_1^2 + A_2^2 + B_2^2. \end{aligned}$$

The multiplicative elliptic functions a , β , λ , δ are Lamé functions of the first order, as they are found to satisfy the differential equations

$$(28) \quad \frac{1}{a} \frac{d^2 a}{du^2}, \quad \text{or} \quad \frac{1}{\delta} \frac{d^2 \delta}{du^2} = 2\wp u + \wp v_1,$$

$$(29) \quad \frac{1}{\beta} \frac{d^2 \beta}{du^2}, \quad \text{or} \quad \frac{1}{\gamma} \frac{d^2 \gamma}{du^2} = 2\wp u + \wp v_2;$$

this follows from their expression by means of (9), (10), (14), (15), in the Weierstrass notation

$$(30) \quad M a = e^{-u\zeta v_1} \frac{\sigma(u + v_1)}{\sigma u \sigma v_1}, \text{ etc.}$$

Also (2) §3 may be written

$$(31) \quad 2 A_1 n^2 a \gamma = \frac{d}{dt} (\rho e^{wi}).$$

32. With lower rosettes

$$(1) \quad L' + L = 0, \quad G' + G = 0, \quad \phi + \psi = 0, \quad f_1 = 1;$$

$$(2) \quad a = \delta = \cos \frac{1}{2} \theta,$$

$$(3) \quad \beta = i \sin \frac{1}{2} \theta e^{\psi i}, \quad \gamma = i \sin \frac{1}{2} \theta e^{-\psi i}.$$

With upper rosettes or intermediate paths,

$$(4) \quad L' - L = 0, \quad G' - G = 0, \quad \phi - \psi = 0, \quad f_2 = 1, \text{ or } 0,$$

$$(5) \quad \beta = \gamma = i \sin \frac{1}{2} \theta,$$

$$(6) \quad a = \cos \frac{1}{2} \theta e^{\psi i}, \quad \delta = \cos \frac{1}{2} \theta e^{-\psi i}.$$

With cusps on the circle $z = z_2$ or z_1 , writing

$$(7) \quad \begin{aligned} Z &= (1 - z^2)(D - z) - \frac{1}{2} \left(\frac{G - G'z}{An} \right)^2 \\ &= (1 - z^2)(D - z) - 2 \left(\frac{L - L'z}{M} \right)^2, \end{aligned}$$

then $G - G'z$ and $D - z$ are factors of Z ,

$$(8) \quad \frac{G}{G'} = \frac{L}{L'} = D = z_2 \text{ or } z_1,$$

$$(9) \quad w = \omega_2 \text{ or } \omega_1, \quad f_1 + f_2 = 1 \text{ or } 0.$$

Now in the associated curve (ρ', ω') of H in the plane HKC'

$$(10) \quad \omega' = p't, \quad \phi = pt - \chi';$$

$$(11) \quad \frac{1}{2}M^2 \sin \theta \, e^{xi} = L' \sqrt{x'} + i \sqrt{(x'_3 - x')(x' - x'_1)}, \quad x' = \frac{1}{2}M^2(D - z),$$

and the spherical curve of C' is rectifiable, in the form

$$(12) \quad \frac{s}{c} = \int_{z_3}^z \frac{dz}{\sqrt{(z_1 - z)(z - z_3)}} = 2 \tan^{-1} \sqrt{\frac{z - z_3}{z_1 - z}} \\ = 2 \tan^{-1}(\operatorname{cn} mt) \text{ or } 2 \operatorname{am}(K' - mt),$$

when z_1 and z_2 change places.

33. When f_2 is a rational fraction, so that μv_2 is congruent to a period, we can utilize the preceding results, and express $(A_2 + B_2 i)$ by algebraical functions of z ; and now the μ th powers of β and γ are algebraical functions qualified by exponential functions of the time.

If we wish to construct a case of motion in which α and δ are quasi-algebraical at the same time it will be sufficient for illustration to take $f_1 = 2f_2$, so that $v_1 = 2v_2$, and now $I(2v)$ is deduced very easily from $I(v)$.

As an illustration, take

$$(1) \quad v_1 = \frac{1}{2}\omega_3, \quad v_2 = \omega_1 + \frac{1}{4}\omega_3,$$

so that from (2) §18, (4) §27,

$$(2) \quad M\gamma = i e^{p_2 t i} \left[\left\{ \frac{1}{2}(1 + a)^2 - x \right\} \frac{1}{2} \sqrt{X_1} + i \left\{ \frac{1}{2}(1 + a)^2 + x \right\} \sqrt{\frac{1}{2} X_2} \right]^{\frac{1}{2}},$$

$$(3) \quad M^2 \sin^2 \frac{1}{2} \theta = \frac{(1 + a)^3(1 - a)}{4a} - x^2,$$

$$(4) \quad Ma = p^{p_2 t i} \left[\sqrt{\frac{1}{2} X_1} + i \sqrt{\frac{1}{2} X_2} \right],$$

$$(5) \quad M^2 \cos^2 \frac{1}{2} \theta = k + x^2,$$

$$(6) \quad X_1, X_2 = (1 \pm x)(k \pm x), \quad k = \frac{1}{4} \left(\frac{1}{a} - a \right)^2,$$

$$(7) \quad M^2 = \frac{(1 + a)^3(1 - a)}{4a} + k = \frac{(1 + a)(1 - a^4)}{4a^2}.$$

With

$$(8) \quad v_1 = \frac{2}{3}\omega_3, \quad v_2 = \omega_1 + \frac{1}{3}\omega_3, \quad w = \omega_1 + \omega_3,$$

there are cusps on the circle $z = z_2$; and from (4) §19,

$$(9) \quad M\gamma = i e^{p_2 i} [\tfrac{1}{2}(9 - b^2)\sqrt{(x_3 - x)(x - x_1)} + i(x_3 + 9 - b^2)\sqrt{(x - x_2)}]^\dagger,$$

$$(10) \quad P(v_2) = \tfrac{1}{6}(9 - b^2), \quad M^2\beta\gamma = s - s(v_2) = -x;$$

$$(11) \quad Ma = e^{p_1 i} [\tfrac{1}{2}(b^2 + 3)x - \tfrac{1}{2}(b^2 - 1)(b^2 + 7) + i\sqrt{\tfrac{1}{2}X}]^\dagger,$$

$$(12) \quad P(v_2) = \tfrac{1}{6}(b^2 + 3), \quad M^2a\delta = 2(b^2 - 1) - x, \quad M^2 = 2(b^2 - 1).$$

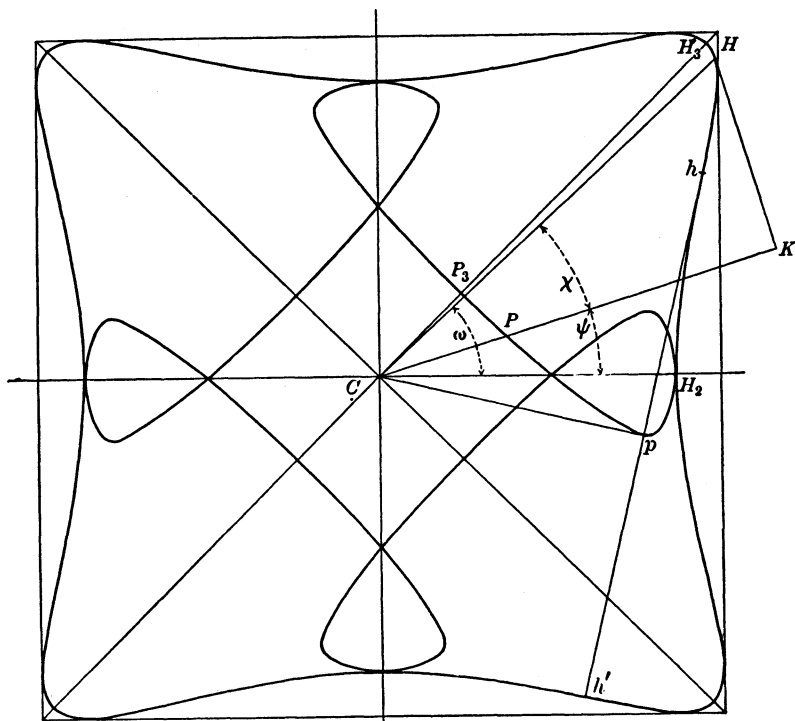


FIG. 5.

With (L.M.S., vol. 25, p. 239 and the results of §24),

$$(13) \quad v_1 = \tfrac{2}{3}\omega_3, \quad v_2 = \omega_1 + \tfrac{1}{3}\omega_3,$$

$$(14) \quad M\gamma = i e^{p_2 i} [(P_1x + P_2)\sqrt{(x_3 - x)(x - x_1)} + i(x^2 + Q_1x + Q_2)\sqrt{(x - x_2)}]^\dagger,$$

$$(15) \quad P(v_2) = \tfrac{1}{3}(c + 3)(-c^2 + 4c + 1), \quad M^2\beta\gamma = -x;$$

$$(16) \quad Ma = e^{p,ti} [R_0 x^2 + R_1 x + R_2 + i\{x - 4(c-1)(c^3 + 7c^2 - c + 1)\sqrt{\frac{1}{4}X}\}]^{\frac{1}{2}},$$

$$(17) \quad P(v_1) = \frac{1}{2}(c^3 - c^2 + 7c - 3), \quad M^2 a \delta = 8c(c+1)^2(c-1) - x_1, \\ M^2 = 8c(c+1)^2(c-1);$$

$$(18) \quad R_0 = c^3 - c^2 + 7c - 3, \\ R_1 = 4(c-1)(-2c^5 - 7 - 23 - 46 + 8 + 5 + 1), \\ R_2 = 16c(c+1)^3(c-1)^2(c^5 + 3c^4 + 34c^3 - 2c^2 - 3c - 1).$$

With the original s of the standard form of $I(v)$ in (1) §10,

$$(19) \quad s = 8c(c+1)^2(c-1) - x,$$

$$(20) \quad Ma = e^{p,ti} [(c^3 - c^2 + 7c - 3)s^2 - 4(c-1)^4(2c^3 + 3c^2 + 4c - 1)s \\ + 16c(c-1)^8(c+1)^2 + i\{s - 4(c-1)^4\sqrt{\frac{1}{4}S}\}]^{\frac{1}{2}}$$

$$(21) \quad M^2 a \delta = \frac{1}{2}M^2(1 + \cos \theta) = s;$$

and putting $s = x^2$, we can halve the degree of Ma by writing it

$$(22) \quad Ma = e^{p,ti} [\{x + 2(c-1)^2\sqrt{\frac{1}{2}S_1} + i\{x - 2(c-1)^2\sqrt{\frac{1}{2}S_2}\}]^{\frac{1}{2}}.$$

34. The geometrical interpretation of the herpolhode and associated motion of the top, by means of Poinso's rolling quadric and Darboux's deformable hyperboloid is required for the complete treatment of the problem, but this would lead too far at present. *Nature*, Aug., 1899, may be consulted for a slight sketch.

LONDON, MARCH, 1903.

ADDENDA AND ERRATA.

Page 7. Equation (12) § 7 is equivalent to

$$\frac{OK^2 = \delta^2 - 2\delta\delta' \cos \theta + \delta'^2}{\sin^2 \theta} = \frac{CC'^2}{\sin^2 \theta},$$

as is evident geometrically, so that the characteristic property of the polhode line of curvature depends otherwise, on equation (2), and the constancy of the lengths δ, δ' cut off from the generating lines of the confocal hyperboloid of one sheet through a line of curvature on an ellipsoid, by perpendicular planes through the centre.

Page 18. Line 2, add — "with angular velocity $2nsh \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2$."

Page 18. Line 12, add — "with angular velocity $2nch \frac{1}{2}\theta_1 \cos \frac{1}{2}\theta_2$."

Page 20. Line 2, read " $= L(L^2 + A)^2 - 8L^2Q - 4LB$."